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Interpolation between Sum and Intersection of Banach Spaces

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We study intermediate Banach spaces A which are or are not interpolation spaces between $A_0 + A_1$ and $A_0 \cap A_1$.

1. INTRODUCTION

We recall some notations from interpolation theory.

A pair $\bar{A} = (A_0, A_1)$ of Banach spaces is called a *Banach couple* if A_0 and A_1 are both continuously imbedded in some Hausdorff topological vector space V.

For a Banach couple $\bar{A} = (A_0, A_1)$ we can form the *sum* $\Sigma(\bar{A}) = A_0 + A_1$ and the *intersection* $\Delta(\bar{A}) = A_0 \cap A_1$. They are both Banach spaces, in the natural norms $\|a\|_{\Sigma(\bar{A})} = K(1, a; \bar{A})$ and $\|a\|_{\Delta(\bar{A})} = \max(\|a\|_{A_0}, \|a\|_{A_1})$, respectively (whenever possible we suppress the "unnecessary" $\bar{A}$, writing Σ and Δ), where

$$K(t, a) \equiv K(t, a; \bar{A}) = \inf\{\|a_0\|_{A_0} + t\,\|a_1\|_{A_1} : a_0 \in A_0,\, a_1 \in A_1,\, a = a_0 + a_1\} \tag{1}$$

for any positive number t.

A Banach space A is called an *intermediate space* between A_0 and A_1 (or with respect to $\bar{A}$) if $\Delta(\bar{A}) \subset A \subset \Sigma(\bar{A})$ with continuous inclusions.

We denote by $L(\bar{A})$ the Banach space of all linear operators $T: \Sigma(\bar{A}) \to \Sigma(\bar{A})$ such that the restriction of T to the space A_i is a bounded operator from A_i into A_i, $i = 0, 1$, with the norm

$$\|T\|_{L(\bar{A})} = \max(\|T\|_{A_0 \to A_0}, \|T\|_{A_1 \to A_1}).$$

An intermediate space A is called an *interpolation space* between A_0 and A_1 (or with respect to $\bar{A}$) if in addition every linear operator from $L(\bar{A})$

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maps A into itself. The set of all interpolation spaces between A_0 and A_1 will be denoted by $I(A_0, A_1)$.

The closed graph theorem implies that a mapping $T\colon A \to A$ is bounded linear, and that there exists a positive constant C such that

$$\|T\|_{A \to A} \leqslant C \|T\|_{L(\bar{A})} \tag{2}$$

for any $T \in L(\bar{A})$ (see [6, p. 20]).

Clearly, $A_0 + A_1$ and $A_0 \cap A_1$ are interpolation spaces between A_0 and A_1, and the constant C in (2) is equal to 1.

The plan of the paper is as follows:

In Section 2 we give a short proof of the Aronszajn–Gagliardo theorem, giving necessary and sufficient conditions for a Banach couple of spaces A_0 and A_1 to be interpolation spaces between $A_0 + A_1$ and $A_0 \cap A_1$. The method of proof is similar to that of Aronszajn and Gagliardo. The main difference lies in the consequent use of the K-functional (briefly proved Proposition 1 instead of Lemma 10.X of Aronszajn and Gagliardo).

Applications of the real method and the Caldéron–Lozanovskiĭ construction to interpolation of the sum and the intersection are given in Section 3 and 4.

In Section 5, the above results are applied to the important class of symmetric function spaces, in particular to Lebesgue, Lorentz, and Orlicz spaces. For example, $L_p(0, \infty)$ is an interpolation space between $L_1(0, \infty) + L_\infty(0, \infty)$ and $L_1(0, \infty) \cap L_\infty(0, \infty)$ if and only if $p = 2$.

Conventions. Two Banach spaces A and B are considered as equal ($A = B$) whenever $A = B$ as sets and their norms are equivalent. The equivalence $a \approx b$ means that $c_0 a \leqslant b \leqslant c_1 a$ for some positive constants c_0 and c_1.

2. A Short Proof of Aronszajn–Gagliardo Theorem

For an intermediate space A with respect to $\bar{A}$ we denote by A^0 the closure of $\Delta(\bar{A})$ in A, and by $\bar{A}^\Sigma$ the closure of A in $\Sigma(\bar{A})$. The following result is well known (see [1, Theorem 7.V]): if $A \in I(A_0, A_1)$ then A must satisfy one of the four conditions

$$A = A_0 + A_1, \qquad A_0 \subset A \subset \bar{A}_0^\Sigma = A_0 + A_1^0,$$
$$A_1 \subset A \subset \bar{A}_1^\Sigma = A_0^0 + A_1, \qquad A_0 \cap A_1 \subset A \subset \overline{A_0 \cap A_1}^\Sigma.$$

In particular, if $A_0 \cap A_1$ is closed in A_0 and A_1 then $I(A_0, A_1) = \{A_0 + A_1, A_0, A_1, A_0 \cap A_1\}$ and such a pair $\bar{A} = (A_0, A_1)$ is called *trivial*.

The key result to our discussion in this section is the following:

PROPOSITION 1. *Let A be an intermediate space between A_0 and A_1, and let $a \in \Sigma$. Suppose that there exists a sequence $\{a_n\} \subset \Delta$ such that*

$$\|a_n\|_\Delta = 1, \qquad \|a_n\|_{A_1} \to 0, \qquad \text{and} \qquad \|a_n\|_A \geqslant c \qquad \text{for some } c > 0. \tag{3}$$

Then there exists a sequence $\{T_n\}$ of operators of rank 1 such that

$$\sup_n \|T_n\|_{L(\bar{A})} \leqslant 1 \qquad \text{and} \qquad \limsup_{n \to \infty} \|T_n a\|_A \geqslant c \lim_{t \to \infty} K(t, a; \bar{A}). \tag{4}$$

Proof. Let $t_n^{-1} = \|a_n\|_{A_1}$. Consider the linear operators $T_n x = a_n f_n(x)$, where f_n are bounded linear functionals on Σ with $f_n(a) = K(t_n, a)$ and $|f_n(x)| \leqslant K(t_n, x)$. The existence of such functionals follows from the Hahn–Banach theorem. If $x \in A_i$, by (3)

$$\|T_n x\|_{A_i} = \|a_n\|_{A_i} |f_n(x)| \leqslant \|a_n\|_{A_i} K(t_n, x) \leqslant \|a_n\|_{A_i} t_n^i \|x\|_{A_i} = \|x\|_{A_i}$$

$i = 0, 1$. Hence, by (3)

$$\limsup_{n \to \infty} \|T_n a\|_A = \limsup_{n \to \infty} \|a_n\|_A |f_n(a)| = \limsup_{n \to \infty} \|a_n\|_A K(t_n, a)$$

$$\geqslant c \limsup_{n \to \infty} K(t_n, a) = c \lim_{t \to \infty} K(t, a)$$

and the proof is complete.

Note that if Δ is a non-closed subspace in A_1 and $\Delta \subset A \subset A_0$, then condition (3) holds.

PROPOSITION 2. *If $A_0 \neq \Sigma$, then the set $\tilde{A}_0 = \{a \in \Sigma: \lim_{t \to \infty} K(t, a; \bar{A}) < \infty\}$ is a first category subset of Σ.*

Proof. First we shall prove that

$$A_r = \overline{S_0(r)}^{\Sigma} \tag{5}$$

for all $r > 0$, where $A_r = \{a \in \Sigma: \sup_{t > 0} K(t, a) \leqslant r\}$ and $S_0(r) = \{a \in A_0: \|a\|_{A_0} \leqslant r\}$.

If $a \in \overline{S_0(r)}^{\Sigma}$ then there exist $a_n \in A_0$ such that $\sup \|a_n\|_{A_0} \leqslant r$ and $\lim_{n \to \infty} \|a - a_n\|_\Sigma = 0$. We have $K(t, a) = \lim_{n \to \infty} K(t, a_n) \leqslant \|a_n\|_{A_0} \leqslant r$ for all $t > 0$. Hence $a \in A_r$. Let $a \in A_r$ and $\varepsilon > 0$. We can find a decomposition $a = a_{0n} + a_{1n}$ such that $a_{0n} \in A_0$, $a_{1n} \in A_1$ and

$$\|a_{0n}\|_{A_0} + n \|a_{1n}\|_{A_1} \leqslant K(n, a) + \varepsilon \leqslant r + \varepsilon.$$

It follows that

$$\|a-a_{0n}\|_{\Sigma} \leqslant \|a_{1n}\|_{A_1} \leqslant \frac{r+\varepsilon}{n} \to 0, \qquad n\to\infty$$

$$\|a_{0n}\|_{A_0} \leqslant K(n, a)+\varepsilon \leqslant r+\varepsilon.$$

Thus, $a \in \bigcap_{\varepsilon>0} \overline{S_0(r+\varepsilon)}^{\Sigma} = \overline{S_0(r)}^{\Sigma}$. Since $\tilde{A}_0$ with the norm $\|a\|_{\tilde{A}_0} = \lim_{t\to\infty} K(t, a)$ (this is the Minkowski functional of the set $\overline{S_0(1)}^{\Sigma}$) is a Banach space and $\tilde{A}_0 = \bigcup_{n=1}^{\infty} \overline{S_0(n)}^{\Sigma}$ and $\overline{S_0(n)}^{\Sigma}$ is a nowhere dense set in Σ, we get our conclusion.

The final result of this section is the Aronszajn–Gagliardo theorem.

THEOREM 1. *Suppose that $A_0 \neq \Delta$ and $A_1 \neq \Delta$ (i.e., $A_0 \neq \Delta$ and $A_0 \neq \Sigma$).*

(a) *If Δ is a non-closed subspace in A_i, then $A_{1-i} \notin I(\Sigma, \Delta)$ $(i=0, 1)$.*

(b) *If Δ is closed in A_0 but not in A_1, then $A_1 \in I(\Sigma, \Delta)$ if and only if Δ is dense in A_1*

(c) *If Δ is closed in both A_0 and A_1, then $A_0, A_1 \notin I(\Sigma, \Delta)$.*

Proof. (a) Suppose that $A_{1-i} \in I(\Sigma, \Delta)$. On account of the assumption there exists a sequence $\{a_n\} \subset \Delta$ such that $\|a_n\|_{\Delta}=1$, $\|a_n\|_{A_i} \to 0$. It follows that $\|a_n\|_{\Delta}=1$, $\|a_n\|_{\Sigma} \to 0$, and $\|a_n\|_{A_{1-i}}=1$. Applying Proposition 1 to the couple (Δ, Σ) and the space $A=A_{1-i}$, we obtain that for any $a \in A_{1-i}$

$$c_1 \|a\|_{A_{1-i}} \geqslant \lim_{t\to\infty} K(t, a; \Delta, \Sigma).$$

Hence $A_{1-i} \subset \tilde{\Delta} = \tilde{A}_0 \cap \tilde{A}_1$; this means that $A_{1-i} \subset \tilde{A}_i$. Since $A_i \subset \tilde{A}_i$ we have $\Sigma \subset \tilde{A}_i$ in contradiction to Proposition 2.

(b) Since $\bar{A}_1^{\Sigma} = A_0^0 + A_1 = \Delta + A_1 = A_1$, it follows that A_1 is a closed subspace of Σ. The closed subspaces of Σ from $I(\Sigma, \Delta)$ are only Σ and $\bar{\Delta}^{\Sigma} = A_1^0$. Hence $A_1 \in I(\Sigma, \Delta)$ if and only if Δ is dense in A_1.

(c) In this case $I(\Sigma, \Delta) = \{\Sigma, \Delta\}$.

3. THE K-METHOD FOR THE SUM AND THE INTERSECTION

Let $\mathscr{P}$ denote the set of all positive functions φ on $R_+ = (0, \infty)$ such that both $\varphi(t)$ and $t\varphi(1/t)$ are non-decreasing, i.e., $\varphi(s) \leqslant \max(1, s/t)\, \varphi(t)$ for all $s, t \in R_+$. $\mathscr{P}$ contains all concave functions on R_+. On $\mathscr{P}$ we define the involution by $\varphi^*(t) = t\varphi(1/t)$. A function φ in $\mathscr{P}$ is said to belong to $\mathscr{P}^{+-}$ if $\min(1, 1/t)\, s_{\varphi}(t) \to 0$ as $t \to 0, \infty$, where $s_{\varphi}(t) = \sup_{u>0}(\varphi(ut)/\varphi(u))$.

Let $\varphi \in \mathscr{P}$ and $p=\infty$ or $\varphi \in \mathscr{P}^{+-}$ and $1 \leqslant p < \infty$. We then define the *real interpolation space* $\bar{A}_{\varphi,p}=(A_0, A_1)_{\varphi,p}$ as the space of all $a\in\Sigma(\bar{A})$ such that

$$\|a\|_{\varphi,p}=\left(\int_0^\infty\left(\frac{K(t,a;\bar{A})}{\varphi(t)}\right)^p\frac{dt}{t}\right)^{1/p},\qquad 1\leqslant p<\infty$$
$$=\sup_{0<t<\infty}\frac{K(t,a;\bar{A})}{\varphi(t)},\qquad p=\infty \tag{6}$$

is finite; it is a Banach space. If $\varphi(t)=t^\Theta(0\leqslant\Theta\leqslant 1)$ we write, in short, $\bar{A}_{\Theta,p}$ and $\|\ \|_{\Theta,p}$. See [5, 2] for details.

If $\varphi_0, \varphi_1\in\mathscr{P}$ and $p=\infty$ or if $\varphi_0,\varphi_1\in\mathscr{P}^{+-}$ and $1\leqslant p<\infty$, then

$$\bar{A}_{\min(\varphi_0,\varphi_1),p}=\bar{A}_{\varphi_0,p}\cap\bar{A}_{\varphi_1,p}\qquad\text{and}\qquad \bar{A}_{\max(\varphi_0,\varphi_1),p}=\bar{A}_{\varphi_0,p}+\bar{A}_{\varphi_1,p} \tag{7}$$

(cf.[3, p. 169]).

PROPOSITION 3. (a) *If* $\varphi(t)/\sqrt{t}$ *is a non-increasing function then*

$$(\Sigma(\bar{A}), \Delta(\bar{A}))_{\varphi,p}=\bar{A}_{\varphi,p}+\bar{A}_{\varphi^*,p}.$$

(b) *If* $\varphi(t)/\sqrt{t}$ *is a non-decreasing function then*

$$(\Sigma(\bar{A}), \Delta(\bar{A}))_{\varphi,p}=\bar{A}_{\varphi,p}\cap\bar{A}_{\varphi^*,p}.$$

(c) *If* $\varphi=\varphi^*$ *then* $(\Sigma(\bar{A}), \Delta(\bar{A}))_{\varphi,p}=\bar{A}_{\varphi,p}$.

Proof. We have $K(t,a;\Sigma,\Delta)=\|a\|_\Sigma$ if $t\geqslant 1$, and $K(t,a;\Sigma,\Delta)\approx K(t,a;\bar{A})+tK(t^{-1},a;\bar{A})$ if $0<t<1$ (see [10, Theorem 3]).

Assume that $1\leqslant p<\infty$ and $\varphi(t)/\sqrt{t}$ is non-increasing. Then

$$\|a\|_{\varphi,p}^p\approx\int_0^1\left(\frac{K(t,a;\bar{A})+tK(t^{-1},a;\bar{A})}{\varphi(t)}\right)^p\frac{dt}{t}+\|a\|_\Sigma\int_1^\infty\frac{1}{\varphi(t)^p}\frac{dt}{t}$$
$$\approx\int_0^1\left(\frac{K(t,a;\bar{A})}{\varphi(t)}\right)^p\frac{dt}{t}+\int_1^\infty\left(\frac{K(t,a;\bar{A})}{t\varphi(1/t)}\right)^p\frac{dt}{t}+\|a\|_\Sigma C_\varphi$$
$$\approx\int_0^1\left(\frac{K(t,a;\bar{A})}{\max(\varphi(t),\varphi^*(t))}\right)^p\frac{dt}{t}$$
$$+\int_1^\infty\left(\frac{K(t,a;\bar{A})}{\max(\varphi(t),\varphi^*(t))}\right)^p\frac{dt}{t}+C_\varphi\|a\|_\Sigma$$
$$\approx\|a\|_{\max(\varphi,\varphi^*),p}.$$

Applying (7) we have

$$(\Sigma,\Delta)_{\varphi,p}=\bar{A}_{\max(\varphi,\varphi^*),p}=\bar{A}_{\varphi,p}+\bar{A}_{\varphi^*,p}.$$

The proofs for the remaining cases are analogous to the above and therefore omitted.

COROLLARY 1 (cf. [10]).

$$(\Sigma(\bar{A}), \Delta(\bar{A}))_{\Theta,p} = \bar{A}_{\Theta,p} + \bar{A}_{1-\Theta,p}, \qquad 0 \leqslant \Theta \leqslant \tfrac{1}{2}$$
$$= \bar{A}_{\Theta,p} \cap \bar{A}_{1-\Theta,p}, \qquad \tfrac{1}{2} \leqslant \Theta \leqslant 1.$$

THEOREM 2. *If $\varphi(t)/\sqrt{t}$ is a monotone function on R_+ or $\varphi = \varphi^*$, then $\bar{A}_{\varphi,p} + \bar{A}_{\varphi^*,p}$ and $\bar{A}_{\varphi,p} \cap \bar{A}_{\varphi^*,p}$ are interpolation spaces between $\Sigma(\bar{A})$ and $\Delta(\bar{A})$.*

Proof. It is an immediate consequence of Proposition 3, of the equality $(\Sigma, \Delta)_{\varphi,p} = (\Delta, \Sigma)_{\varphi^*,p}$, and of the definition of the real interpolation method.

PROBLEM 1. Let $0 < \Theta < 1$ and $1 \leqslant p, q \leqslant \infty$. Under which conditions on Θ, p and q, $\bar{A}_{\Theta,p} + \bar{A}_{1-\Theta,q}$ and $\bar{A}_{\Theta,p} \cap \bar{A}_{1-\Theta,q}$ are interpolation spaces between $\Sigma(\bar{A})$ and $\Delta(\bar{A})$?

Theorem 2 gives an affirmative answer for $p = q$ and any Θ.

4. Calderón–Lozanovskiĭ Construction for the Sum and the Intersection

Let (Ω, Σ, μ) be a complete σ-finite measure space and let us denote by $L^0 = L^0(\Omega, \Sigma, \mu)$ the space of all equivalence classes of μ-measurable real valued functions, equipped with the topology of convergence in measure on μ-finite sets. We will say tht a Banach space X is a *Banach function space* (on (Ω, Σ, μ)) if X is a Banach subspace of L^0 satisfying the property that if $x \in X$ and $y \in L^0$ are such that $|y(t)| \leqslant |x(t)|$, μ-a.e. on Ω, then $y \in X$ and $\|y\|_X \leqslant \|x\|_X$. Note that if X_0 and X_1 are any two Banach function spaces (on (Ω, Σ, μ)) then $\bar{X} = (X_0, X_1)$ forms a Banach couple.

Let $\bar{X} = (X_0, X_1)$ be a couple of Banach function spaces and let $\varphi \in \mathscr{P}$. We will consider φ as a function on $R_+ \times R_+$ putting $\varphi(s, t) = s\varphi(t/s)$. We denote by $\varphi(\bar{X}) = \varphi(X_0, X_1)$ the *Calderón–Lozanovskiĭ space* of all $x \in L^0$ such that for some $x_i \in X_i$, $\|x_i\|_{X_i} \leqslant 1$, $i = 0, 1$, and for some $\lambda > \infty$ holds $|x| \leqslant \lambda\varphi(|x_0|, |x_1|)$ μ-a.e. We put $\|x\|_{\varphi(\bar{X})} = \inf \lambda$.

Note that $\varphi(\bar{X})$ is a Banach function space (with equivalent norm) as well as an intermediate space with respect to $\bar{X}$. If in particular we take $\varphi(t) = t^{\Theta}$, $0 < \Theta < 1$, we obtain, in this way, the spaces $X_0^{1-\Theta} X_1^{\Theta}$ introduced by Calderón [4]. The properties of $\varphi(\bar{X})$ have been studied in detail by

Lozanovskiĭ [8]. Ovčinnikov in [11] showed that if $\varphi(\bar{X})=\varphi(\bar{X})''$ then $\varphi(\bar{X})$ is an interpolation space with respect to $\bar{X}$. Analogically we define $\varphi^*(\bar{X})=\varphi^*(X_0, X_1)$, where $\varphi^*(s,t)=s\varphi^*(t/s)=t\varphi(s/t)$ and $\varphi\in\mathscr{P}$.

PROPOSITION 4. (a) *If* $\varphi(t)/\sqrt{t}$ *is a non-increasing function then* $\varphi(\Sigma(\bar{X}), \Delta(\bar{X}))=\varphi(\bar{X})+\varphi^*(\bar{X})$.

(b) *If* $\varphi(t)/\sqrt{t}$ *is a non-decreasing function then* $\varphi(\Sigma(\bar{X}), \Delta(\bar{X}))=\varphi(\bar{X})\cap\varphi^*(\bar{X})$.

(c) *If* $\varphi=\varphi^*$ *then* $\varphi(\Sigma(\bar{X}), \Delta(\bar{X}))=\varphi(\bar{X})$.

Proof. First, we note that if $x_i\in X_i$, $i=0, 1$, then $\min(|x_0|, |x_1|)\in\Delta(\bar{X})$ and $\max(|x_0|, |x_1|)\in\Sigma(\bar{X})$. Moreover, $\|\min(|x_0|, |x_1|)\|_\Delta\leqslant\max(\|x_0\|_{X_0}, \|x_1\|_{X_1})$ and $\|\max(|x_0|, |x_1|)_1\|_\Sigma\leqslant\|x_0\|_{X_0}+\|x_1\|_{X_1}$. Let us denote $X_\varphi=\varphi(\Sigma(\bar{X}), \Delta(\bar{X}))$. Let us first show that $\varphi(\bar{X})\cap\varphi^*(\bar{X})\subset X_\varphi$: Let $|x|\leqslant\varphi(|y_0|, |y_1|)$ and $|x|\leqslant\varphi^*(|z_0|, |z_1|)=\varphi(|z_1|, |z_0|)$ where $\|y_i\|_{X_i}\leqslant 1$, $\|z_i\|_{X_i}\leqslant 1$, $i=0, 1$. Then

$$
\begin{aligned}
|x| &\leqslant \min\{\varphi(|y_0|, |y_1|), \varphi(|z_1|, |z_0|)\}\\
&\leqslant \min\{\varphi(\max(|y_0|, |z_1|), |y_1|), \varphi(\max(|y_0|, |z_1|)), |z_0|)\}\\
&= \varphi(\max(|y_0|, |z_1|), \min(|y_1|, |z_0|)).
\end{aligned}
$$

Hence $\|x\|_{X_\varphi}\leqslant 2\max(\|x\|_{\varphi(\bar{X})}, \|x\|_{\varphi^*(\bar{X})})$. Second, if $\varphi(t)/\sqrt{t}$ is non-decreasing, then $X_\varphi\subset\varphi(\bar{X})\cap\varphi^*(\bar{X})$. Indeed, since $\varphi(t)/\sqrt{t}$ is non-decreasing, we have $\varphi(s,t)\leqslant\varphi(t, \max(s,t))$ for all $s, t\in R_+$. For every $x\in X_\varphi$, there exist $x_i\in X_i$, $y\in\Delta(\bar{X})$ such that $|x|\leqslant\varphi(|x_0+x_1|, |y|)$. Then

$$
\begin{aligned}
|x| &\leqslant \varphi(|x_0|, |y|)+\varphi(|x_1|, |y|)\\
&\leqslant \varphi(\max\{|x_0|, |y|\}, |y|)+\varphi(|y|, \max\{|x_1|, |y|\})
\end{aligned}
$$

and

$$
|x|\leqslant\varphi(|y|, \max\{|x_0|, |y|\})+\varphi(\max\{|x_1|, |y|\}, |y|).
$$

From the first inequality it follows that $x\in\varphi(\bar{X})$ and from the second $x\in\varphi^*(\bar{X})$. Third,

$$
X_\varphi\subset\varphi(\bar{X})+\varphi^*(\bar{X}).
$$

This follows directly from the inequality

$$
\varphi(|x_0+x_1|, |y|)\leqslant\varphi(|x_0|, |y|)+\varphi(|x_1|, |y|).
$$

Fourth, if $\varphi(t)/\sqrt{t}$ is non-increasing, then $\varphi(\bar{X})+\varphi^*(\bar{X})\subset X_\varphi$. In fact, since $\varphi(t)/\sqrt{t}$ is non-increasing, it follows that $\varphi(s,t)\leqslant\varphi(\max\{s,t\},$

$\min\{s, t\})$ for all $s, t \in R_+$. Let $x = x_0 + x_1$, where $x_0 \in \varphi(\bar{X})$ and $x_1 \in \varphi^*(\bar{X})$. There exist $y_0, z_0 \in X_0$ and $y_1, z_1 \in X_1$ such that $|x_0| \leqslant \varphi(|y_0|, |y_1|)$, $|x_1| \leqslant \varphi^*(|z_0|, |z_1|) = \varphi(|z_1|, |z_0|)$. Then

$$\begin{aligned} |x| &\leqslant |x_0| + |x_1| \leqslant (|y_0|, |y_1|) + \varphi(|z_1|, |z_0|) \\ &\leqslant \varphi(\max\{|y_0|, |y_1|\}, \min\{|y_0|, |y_1|\}) \\ &\quad + \varphi(\max\{|z_0|, |z_1|\}, \min\{|z_0|, |z_1)\}) \end{aligned}$$

and we conclude that $x \in X_\varphi$. Part (c) can be proved in a similar way. Thus our proposition is proved.

Corollary 2 (Semenov–Šneiberg; see [13, Theorem 3]).

$$\begin{aligned} (X_0 + X_1)^{1-\Theta}(X_0 \cap X_1)^{\Theta} &= X_0^{1-\Theta} X_1^{\Theta} + X_0^{\Theta} X_1^{1-\Theta}, \qquad 0 \leqslant \Theta \leqslant \tfrac{1}{2} \\ &= X_0^{1-\Theta} X_1^{\Theta} \cap X_0^{\Theta} X_1^{1-\Theta}, \qquad \tfrac{1}{2} \leqslant \Theta \leqslant 1. \end{aligned}$$

From Ovčinnikov's interpolation theorem and Proposition 4 we get:

Theorem 3. *Let $\varphi(\Sigma, \Delta) = \varphi(\Sigma, \Delta)''$. If $\varphi(t)/\sqrt{t}$ is a monotone function on R_+ or $\varphi = \varphi^*$, then $\varphi(\bar{X}) + \varphi^*(\bar{X})$ and $\varphi(\bar{X}) \cap \varphi^*(\bar{X})$ are interpolation spaces between $\Sigma(\bar{X})$ and $\Delta(\bar{X})$.*

Corollary 2 and Theorem 3 suggest the following problems:

Problem 2. Does the result of type of Corollary 2 hold for the complex interpolation method and any Banach spaces?

Problem 3. Is the assumption $\varphi(\Sigma, \Delta) = \varphi(\Sigma, \Delta)''$ necessary in Theorem 3?

5. Concrete Examples

Let $R_+ = (0, \infty)$ be equipped with Lebesgue measure. A Banach function space $E = E(0, \infty)$ is said to be a *symmetric space* (on R_+) if $x \in E$ and $y \in L^0$ and $|y|$ is equimeasurable with $|x|$, then $y \in E$ and $\|y\|_E = \|x\|_E$.

Any non-trivial symmetric space E is intermediate (not necessarily interpolation) between L_1 and L_∞. The *fundamental function* $\varphi = \varphi_E$ of a symmetric space E on $(0, \infty)$ is defined for $t > 0$ as $\varphi_E(t) = \|1_{(0,t)}\|_E$, where $1_{(0,t)}$ is the characteristic function of the interval $(0, t)$.

The sum $\Sigma(\bar{E})$ and the intersection $\Delta(\bar{E})$ of two symmetric spaces E_0, E_1 are also symmetric spaces, and

$$\varphi_{\Sigma(\bar{E})} = \min(\varphi_0, \varphi_1), \qquad \varphi_{\Delta(\bar{E})} = \max(\varphi_0, \varphi_1). \tag{8}$$

If $L_1 \cap L_\infty$ is dense in a symmetric space E, then E is *minimal*, i.e., E does not contain any non-trivial closed symmetric subspace. If $E = E''$, then E is a *maximal* symmetric space, i.e., E is not a proper closed subspace of a some symmetric space. Comprehensive information about symmetric spaces can be found in books [6, 7].

Let E_0, E_1, and E be symmetric spaces on $(0, \infty)$ with the fundamental functions φ_0, φ_1, and φ, respectively. Put

$$\varphi_{01}(t) = \varphi_0(t)/\varphi_1(t).$$

We first describe a necessary condition for the interpolation of symmetric spaces. For a more general result, see [9].

Consider the family of linear operators $\{T_{s,t}\}$ from E into E defined by

$$T_{s,t}x(u) = \left(s^{-1}\int_0^s x(v)\,dv\right) 1_{(0,t)}(u) \qquad (s, t > 0).$$

Then

$$\begin{aligned}\|T_{s,t}x\|_E &= \left|s^{-1}\int_0^s x(v)\,dv\right| \|1_{(0,t)}\|_E \\ &= s^{-1}\left|\int_0^\infty x(v)\,1_{(0,s)}(v)\,dv\right| \varphi(t) \\ &\leqslant s^{-1}\,\|x\|_E\,\|1_{(0,s)}\|_{E'}\,\varphi(t) \\ &= \frac{\varphi(t)}{\varphi(s)}\,\|x\|_E\end{aligned}$$

with equality for $x = 1_{(0,s)}$. Hence, $\|T_{s,t}\|_{E \to E} = \varphi(t)/\varphi(s)$. From the above and (2) we have a necessary condition for interpolation of symmetric spaces. If E is an interpolation space between E_0 and E_1 then there exists a positive constant C such that the following inequality

$$\frac{\varphi(t)}{\varphi(s)} \leqslant C \max\left\{\frac{\varphi_0(t)}{\varphi_0(s)}, \frac{\varphi_1(t)}{\varphi_1(s)}\right\} \qquad \forall s, t > 0. \tag{9}$$

holds.

THEOREM 4. *Let $E_0 \neq E_0 \cap E_1$ and $E_1 \neq E_0 \cap E_1$. If both E_0 and E_1 are separable or $E_0 = E_0''$ and $E_1 = E_1''$ or $\varphi_{01}(R_+) = R_+$, then E_0 and E_1 are not interpolation spaces between $E_0 + E_1$ and $E_0 \cap E_1$.*

Proof. (1°) If both E_0 and E_1 are separable, then $E_0 \cap E_1$ is non-closed in E_0 and E_1. Hence, by Theorem 1(a) we have $E_0, E_1 \notin I(\Sigma, \Delta)$.

(2°) If $E_i = E_i''$, $i = 0, 1$, then both E_0 and E_1 are non-closed in $E_0 + E_1$. Hence $E_0 \cap E_1$ is non-closed in E_0 and E_1. Theorem 1(a) implies that $E_0, E_1 \notin I(\Sigma, \Delta)$.

(3°) Assume that $E_0 \in I(\Sigma, \Delta)$. Since equality (8) holds if follows from (9) that

$$1 \leqslant C \max \left\{ \frac{\min(1, \varphi_{10}(t))}{\min(1, \varphi_{10}(s))}, \frac{\max(1, \varphi_{10}(t))}{\max(1, \varphi_{10}(s))} \right\}$$

for all $s, t > 0$. Taking s_n and t_n such that $\varphi_{10}(s_n) \to \infty$ and $\varphi_{10}(t_n) \to 0$ as $n \to \infty$ we thus have a contradiction. The proof for E_1 is similar.

Now, we solve a question posed by E. M. Semenov showing that there exists a pair of symmetric spaces (E_0, E_1) on $(0, \infty)$ such that $E_0 \neq E_0 \cap E_1$, $E_1 \neq E_0 \cap E_1$, and E_1 is an interpolation space between $E_0 + E_1$ and $E_0 \cap E_1$.

EXAMPLE 1. Let both E_2 and E_3 be non-separable symmetric spaces on $(0, \infty)$, for example: non-separable Orlicz spaces L_M and L_N or non-separable Orlicz and Marcinkiewicz spaces L_M and $M(\varphi)$, respectively. We denote by E_i^0 $(i = 2, 3)$ either the closure of $L_1 \cap L_\infty$ in E_i or a subspace of E_i with absolutely continuous norm. Suppose that $E_2^0 \cap E_3^0$ is not equal to $\{0\}$ or E_2^0, or $E_2 \cap E_3^0$. Put $E_0 = E_2 \cap E_3^0$, $E_1 = E_2^0$. Then $E_0 \cap E_1 = E_2^0 \cap E_3^0$ is closed in E_0 and it is dense in E_1. By Theorem 1(b) we have that $E_1 \in I(E_0 + E_1, E_0 \cap E_1)$.

Let us finally give some examples of the scope of our results.

Note that Theorems 2 and 3 actually yield:

EXAMPLE 2. If $1 \leqslant p,\ q \leqslant \infty$ and $1/p + 1/p' = 1$, then $L_{pq}(0, \infty) + L_{p'q}(0, \infty)$, $L_{pq}(0, \infty) \cap L_{p'q}(0, \infty)$ and $L_p(0, \infty) + L_{p'}(0, \infty)$, $L_p(0, \infty) \cap L_{p'}(0, \infty)$ are interpolation spaces between $L_1(0, \infty) + L_\infty(0, \infty)$ and $L_1(0, \infty) \cap L_\infty(0, \infty)$.

From Theorem 3 and (9) we get the following consequence.

EXAMPLE 3. Let $1 \leqslant p_0 \leqslant p,\ q \leqslant p_1 \leqslant \infty$. The following conditions are equivalent:

(i) $L_p(0, \infty) + L_q(0, \infty) \in I(L_{p_0}(0, \infty) + L_{p_1}(0, \infty) L_{p_0}(0, \infty) \cap L_{p_1}(0, \infty))$,

(ii) $L_p(0, \infty) \cap L_q(0, \infty) \in I(L_{p_0}(0, \infty) + L_{p_1}(0, \infty), L_{p_0}(0, \infty) \cap L_{p_1}(0, \infty))$,

(iii) $1/p + 1/q = 1/p_0 + 1/p_1$.

Proof. Implication (i) or (ii) $\Rightarrow$ (iii) follows from (9). Now we will show the implication (iii) $\Rightarrow$ (i) and (ii).

Define Θ by $1/p=(1-\Theta)/p_0+\Theta/p_1$. Then $1/q=\Theta/p_0+(1-\Theta)/p_1$.

Since $L_{p_0}^{1-\Theta}L_{p_1}^{\Theta}=L_p$ and $L_{p_0}^{\Theta}L_{p_1}^{1-\Theta}=L_q$, by Theorem 3 the implication holds.

In particular, $L_p(0,\infty)$ is an interpolation space between $L_1(0,\infty)+L_\infty(0,\infty)$ and $L_1(0,\infty)\cap L_\infty(0,\infty)$ if and only if $p=2$.

PROBLEM 4. Let $1<p<\infty$ and $1/p+1/p'=1$. Can Orlicz spaces $L_p(0,\infty)+L_{p'}(0,\infty)$ and $L_p(0,\infty)\cap L_{p'}(0,\infty)$ be obtained by the K-method from $L_1(0,\infty)+L_\infty(0,\infty)$ and $L_1(0,\infty)\cap L_\infty(0,\infty)$?

In the next example we apply Theorem 3 to Orlicz spaces.

EXAMPLE 4. Let $M(u)/u^2$ be a monotone function on R_+ and let $N^{-1}(u)=uM^{-1}(1/u)$ for $u\in R_+$, where M^{-1}, N^{-1} are the right continuous inverses of the Orlicz functions M and N, respectively. Then the Orlicz spaces L_M+L_N and $L_M\cap L_N$ are interpolation spaces between L_1+L_∞ and $L_1\cap L_\infty$.

Proof. It is sufficient to prove that if $\varphi(t)=tM^{-1}(1/t)$ then we have $\varphi(L_1,L_\infty)=L_M$. Indeed, if $x\in L_M$ and $\int M(|x|/\lambda)\,dt\leqslant 1$, then for $y=M(|x|/\lambda)$ holds $|x|\leqslant\lambda M^{-1}(M(|x|/\lambda))=\lambda\varphi(y,1)$. Since $\|y\|_{L_1}\leqslant 1$, it follows that $x\in\varphi(L_1,L_\infty)$. Assume conversely that $|x|\leqslant\lambda\varphi(|x_0|,|x_1|)$, where $\|x_0\|_{L_1}\leqslant 1$ and $\|x_1\|_{L_\infty}\leqslant 1$. Then

$$M(|x|/\lambda)\leqslant M(\varphi(|x_0|,|x_1|))\leqslant M(\varphi(|x_0|,1))=M(M^{-1}(|x_0|))\leqslant|x_0|.$$

Hence, $\int M(|x|/\lambda)\,dt\leqslant\int|x_0|\,dt=\|x_0\|_{L_1}\leqslant 1$ and we conclude that $x\in L_M$. Moreover,

$$\begin{aligned}\|x\|_{\varphi(L_1,L_\infty)}&=\inf\{\lambda>0\colon |x|\leqslant\lambda\varphi(|x_0|,|x_1|);\ \|x_0\|_{L_1}\leqslant 1,\ \|x_1\|_{L_\infty}\leqslant 1\}\\&=\inf\{\lambda>0\colon |x|\leqslant\lambda\varphi(|x_0|,1),\ \|x_0\|_{L_1}\leqslant 1\}\\&=\inf\{\lambda>0\colon |x|\leqslant\lambda M^{-1}(|x_0|),\ \|x_0\|_{L_1}\leqslant 1\}\\&=\inf\{\lambda>0\colon M(|x|/\lambda)\leqslant|x_0|,\ \|x_0\|_{L_1}\leqslant 1\}\\&=\inf\{\lambda>0\colon \|M(|x|/\lambda)\|_{L_1}\leqslant 1\}=\|x\|_{L_M}.\end{aligned}$$

Since $\varphi^*(L_1,L_\infty)=L_N$, Theorem 3 now implies that L_M+L_N and $L_M\cap L_N$ are interpolation spaces between L_1+L_∞ and $L_1\cap L_\infty$.

Clearly, for some M, $L_M(0,\infty)+L'_M(0,\infty)$ and $L_M(0,\infty)\cap L'_M(0,\infty)$ are not interpolation spaces between $L_1(0,\infty)+L_\infty(0,\infty)$ and $L_1(0,\infty)\cap L_\infty(0,\infty)$. Namely, for $L_M(0,\infty)=L_2(0,\infty)\cap L_3(0,\infty)$ condition (9) does not hold.

There arises Problem 5 of describing all symmetric spaces that are interpolation spaces between $L_1(0, \infty)+L_\infty(0, \infty)$ and $L_1(0, \infty)\cap L_\infty(0, \infty)$. The answer to this question is open. Ovčinnikov proved in [12] that not all interpolation spaces can be obtain by the K-method.

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